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Some Finitely Generated, Infinitely Related Metabelian Groups with Trivial Multiplicator

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In this article we study the multiplicator and the number of relators of some metabelian groups that arise from specially simple one-relator groups by the process of metabelianization. This leads, in particular, to the seemingly simplest example of a finitely generated, but infinitely related group with trivial multiplicator.

1. INTRODUCTION

The Introduction consists of two parts. In the first part, we give a survey of results dealing with the multiplicator and the number of relators of a finitely generated group. In the second part, we propose a recipe for constructing uncomplicated infinitely related groups and discuss various methods of proving that a given group is actually infinitely related.

1.1. Let G denote a finitely generated group and consider a free presentation

$$R \lhd F \twoheadrightarrow G$$

of G, where F is also finitely generated. The abelianized kernel R_{ab} acquires the structure of a left G-module via conjugation in F, and is called a relation module of G.

If R is the normal closure of a finite number of elements, R_{ab} and also the multiplicator H_2G inherit a finiteness property according to the following chain of implications:

(*) $R \lhd F \twoheadrightarrow G$ is a finite presentation of G.

$\Downarrow$

R_{ab} is a finitely generated G-module.

$\Downarrow$

The multiplicator H_2G of G is finitely generated.

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The first claim is plain, while the second may be derived from the fact that H_2G is a subgroup of $R/[R,F]$ (Schur–Hopf). The converse of the second assertion is false, while it seems an open question whether the converse of the first implication is true.

Remark. If we restrict our attention to free presentations $R \lhd F \twoheadrightarrow G$, where F is also finitely generated, the finiteness properties are stable in the following sense: Either all the kernels R are the normal closure of finitely many elements, or none of them are [11, p. 124, Corollary (12)]. Similarly, either all the relation modules are finitely generated, or none of them. (Apply Schanuel's Lemma to Lyndon's presentation $R_{\mathrm{ab}} \rightarrowtail \mathbb{Z}G \otimes_F IF \twoheadrightarrow IG$ ([9], cf. [8], p. 199, Corollary 6.4).

1.2. We next survey the behavior of finitely generated groups under extensions.

(**) The properties "has a finite presentation," "has a finitely generated relation module," and "H_2 is finitely generated" are preserved under extensions.

The first result is due to Hall [7, p. 426, Lemma 1], the second is implicit in a paper by Bieri–Eckmann [5, p. 81 Proposition 3.3(a)], whereas the last statement is guaranteed by spectral sequence theory (by hypothesis all groups under consideration are finitely generated).

1.3. If we pass on to the connection between a group and a subgroup of finite index, a surprise occurs. For, we have:

(***) Let U be a subgroup of finite index in G, and assume G is finitely generated. Then:

(a) U is finitely related $\Leftrightarrow$ G is finitely related.

(b) U has a finitely generated relation module $\Leftrightarrow$ G has a finitely generated relation module.

(c) H_2U is finitely generated $\Rightarrow$ H_2G is finitely generated.

Claim (a) follows from the Reidemeister–Schreier method for presenting subgroups, and from (**). Claim (b) is readily deduced from [5, Proposition 1.2] (see also Criterion E in Subsection 1.10). A sketch of a proof of (c) is as follows: Present G as F/R with F finitely generated, and U as S/R. Note that S is also finitely generated. Then, compute H_2U from $R \cap S'/[R, S]$ and H_2G from $R \cap F'/[R, F]$.

The converse of (c) is false. In [1], Baumslag constructs a finitely generated group having cyclic multiplicator and possessing a normal subgroup N of finite index whose multiplicator H_2N is not finitely generated.

1.4. We turn now to the second part of the Introduction. We first concern ourselves with a recipe for constructing uncomplicated infinitely related groups. Suppose

$$G = \langle x_1, \ldots, x_p : r_1, \ldots, r_q \rangle,$$

is a finitely presented group and consider $G/\mathfrak{v}(G)$, the "image of G in the variety $\mathfrak{v}$". Even in so simple a case as the variety of all metabelian groups, the effect may be quite drastic. Indeed we have:

THEOREM (Baumslag [3]). *If $p - q \geqslant 2$, then the multiplicator of G/G'' is not finitely generated (and hence, G/G'' is not finitely related).*

For $p - q < 2$, anything can happen, as the next theorem, the main result of this paper, indicates:

THEOREM. *Let G be the one-relator group*

$$\langle a, t \,:\, t^r \cdot a^m \cdot t^{-r} = a^n \rangle, \qquad mnr \neq 0,$$

and denote by g.c.d. (m, n) *the greatest common divisor of m and n. Then:*

(i) *If either* (a) g.c.d. $(m, n) > 1$, *or* (b) $|r| > 1$ *and* $mn = 1$, *then* $H_2(G/G'')$ *is not finitely generated, and therefore, G/G'' is not finitely related.*

(ii) *If either* (a) $|r| = 1$ *and* $|m|$ *or* $|n| = 1$, *or* (b) $|r| > 1$ *and* $|m|$ *or* $|n| = 1$, *and* $m \neq n$, *then G/G'' is finitely related.*

(iii) *If* g.c.d. $(m, n) = 1$ *and* $|m| \neq 1 \neq |n|$, *then G/G'' is not finitely related, although $H_2(G/G'')$ is finitely generated.*

Finally, even though $H_2(G/G'')$ is finitely generated in (iii), *none of the relation modules of G/G'' is finitely generated (as a $Z(G/G'')$-module).*

1.5. Concluding the Introduction, we now discuss three methods that seem to be at hand if one tries to show that a given group is not finitely related. During the discussion, we shall explain how the various cases of our main result are proved.

The first method, namely, the investigation of the multiplicator H_2Q of Q, is probably the easiest one. If H_2Q is infinitely generated, Q is not finitely related (see (*)). Case (i) of our theorem, e.g., is settled in this way. If H_2Q is finitely generated, the second method may be used. This is the direct approach and poses two problems. The first problem is to present the group explicitly by a specific set of generators, say

$$x_1, \ldots, x_p,$$

and a specific infinite set of relators, say

$$r_1, r_2, \ldots, r_j, \ldots, \qquad (1 \leqslant j < \omega).$$

The second problem is to prove that no finite subset

$$r_{i_1}, r_{i_2}, \ldots, r_{i_q}$$

of the above sequence of relators suffices to define Q.

1.6. In Case (iii) of our theorem, we resort to the second method. Since it is not easy to present G/G'' in a tractable manner, we pass to a quotient of G/G''. Let N denote the normal closure of the generator a in G. This normal subgroup contains the commutator subgroup G' of G, and thus, leads to a canonical surjection

$$p : G/G'' \twoheadrightarrow G/N'.$$

Since later on we have to find our way back to G/G'', we must know what the kernel of p is like. It follows from the main theorem of Hall [7] that ker p is finitely generated as a normal subgroup of G/G''. This provides us with a way to prove that G/G'' is not finitely related in Case (iii).

Remark. If m and n are relatively prime, and $mn \neq 1$, then ker p is a free abelian group of finite rank. This result is proved in Lemma A (Section 3) and will be important for the discussion of Case (ii) and of the multiplicator in Case (iii).

In contradistinction to G/G'', the quotient group G/N' admits a perspicuous presentation, namely,

$$\langle a, t : (a^m)^{t^r} = a^n, [a, a^t], [a, a^{t^2}], \ldots, [a, a^{t^j}], \ldots \rangle.$$

We still face the second task, requiring a proof that for no natural number q the presentation

$$\langle a, t : (a^m)^{t^r} = a^n, [a, a^t], [a, a^{t^2}], \ldots, [a, a^{t^q}] \rangle \tag{1.1}$$

defines the given group. It is possible to prove, using generalized free products, that for every q the group (1.1) contains noncyclic free subgroups (provided of course that neither $|m|$ nor $|n|$ equals 1). We shall, however, give a different argument which we motivate below.

1.7. Let L be a finitely generated nonhopfian group, and let $\vartheta: L \to L$ be a homomorphism of L onto L with nontrivial kernel. Consider the chain

$$L_0 \xrightarrow{\vartheta} L_1 \xrightarrow{\vartheta} L_2 \xrightarrow{\vartheta} \cdots \xrightarrow{\vartheta} L_j \xrightarrow{\vartheta} \cdots, \tag{1.2}$$

where $L_j = \vartheta^j(L)$ $(=L)$ for every $j < \omega$. View (1.2) as a diagram over the directed set ω and pass to the colimit L_ω. It follows easily that

$$L_\omega \cong L \Big/ \bigcup_{j<\omega} \ker(\vartheta^j).$$

Since $\ker(\vartheta^j)$ is properly contained in $\ker(\vartheta^{j+1})$ for every $j < \omega$, the colimit L_ω is not finitely related.

Now, assume $\vartheta: L \to L$ induces an isomorphism

$$\vartheta_* : H_2L \cong H_2L.$$

Since the functor H_2 commutes with direct limits, we obtain an isomorphism

$$H_2L \cong H_2L_\omega .$$

If, in particular, H_2L is finitely generated, H_2L_ω will also be finitely generated, although L_ω is infinitely related.

1.8. As an illustration, consider

$$L = \langle a, t : (a^2)^t = a^3 \rangle.$$

The homomorphism $\vartheta : L \to L$, given by

$$a \mapsto a^2, \qquad t \mapsto t,$$

is onto, but not injective (Baumslag–Solitar [4]). The kernel of ϑ is the normal closure of

$$[a, a^t]$$

in L (see the remarks at the end of Section 4). The limit group L_ω has the presentation

$$\langle a, t : (a^2)^t = a^3, [a, a^t], [a, a^{t^2}], \ldots, [a, a^{t^j}], \ldots \rangle,$$

and thus, coincides with L/L''. But $H_2L = 0$. Hence, L/L'' has trivial multiplicator, although it is not finitely related. It is worth noting that L/L'' is the split extension of Q_6, the additive group of rational numbers with denominator a power of 6, by an infinite cyclic group that acts on Q_6 by multiplication by $3/2$.

1.9. We indicate now how Cases (ii) and (iii) of our main result are proved. Case (ii) and that part of Case (iii) dealing with the number of relators of G/G'', are both taken care of by the next lemma. It is proved in Section 4.

LEMMA C. *Let G be the one-relator group*

$$\langle a, t : (a^m)^{t^r} = a^n \rangle, \qquad mnr \neq 0,$$

and let N denote the normal closure of a in G. If m and n are relatively prime, and $mn \neq 1$, *the following statements are equivalent:*

(a) $|m|$ *or* $|n| = 1$.

(b) $\langle a, t : (a^m)^{t^r} = a^n, [a, a^t], [a, a^{t^2}], \ldots, [a, a^{t^{|r|-1}}] \rangle$ *is metabelian.*

(c) G/N' *is finitely related.*

(d) G/G'' *is finitely related.*

We mention that the proof of the implication "not (b) $\Rightarrow$ not (c)" is based on ideas introduced in Subsection 1.7.

The other assertion in Case (iii) is established in Section 5. This proof too relies on ideas in Subsection 1.7.

1.10. The third method appears to be the hardest one to carry through. It requires a proof that no relation module of the given group Q is finitely generated. By (*), this suffices to ensure that Q is not finitely related.

It is worth pointing out that the question whether Q has a finitely generated relation module, can be phrased in an alternative way.

CRITERION E (Bieri–Eckmann [5]). *Let* $S \lhd F \twoheadrightarrow Q$ *be a free presentation of the finitely generated group* Q, *the free group* F *being of finite rank. Then, the relation module* S_{ab} *is finitely generated if and only if*

$$H_1\Bigl(Q, \prod_{j<\omega} (\mathbb{Z}Q)_j\Bigr),$$

the first homology group of Q *with coefficients in the Cartesian power* $\prod_{j<\omega}(\mathbb{Z}Q)_j$ *of the integral group ring* $\mathbb{Z}Q$, *is trivial.*

If Q has an uncomplicated structure, it may actually be decidable whether

$$H_1\Bigl(Q, \prod_{j<\omega} (\mathbb{Z}Q)_j\Bigr) \neq 0.$$

For the groups of our theorem, this is indeed possible [12]. In case (iii), e.g., one can exhibit a nontrivial element in $H_1(G/G'', \prod \mathbb{Z}(G/G''))$, which proves G/G'' does not admit a finitely generated relation module.

1.11. *Historical remark.* The group $L = G/G''$, where

$$G = \langle a, t : (a^2)^t = a^3 \rangle,$$

was brought to our attention in May 1973 by Bieri, who asked whether L is infinitely related. Also at about that time, Conway asked Higman the same question. Higman's answer was affirmative.

2. The Proof of Case (i)

2.1. This case is easy and rests upon the following observations.

A finitely generated, metabelian group satisfies the maximal condition for normal subgroups (Hall [7, p. 430, Theorem 3]). As a consequence, any homomorphic image of a finitely related metabelian group, is finitely related.

An analogous consequence holds for the multiplicator (Baumslag [1, p. 240, Lemma 3]). It says that any quotient of a finitely generated metabelian group with finitely generated multiplicator, has a finitely generated multiplicator.

2.2. Consider now

$$G = \langle a, t : (a^m)^{t^r} = a^n \rangle, \qquad mnr \neq 0.$$

In Case (i), either g.c.d. $(m, n) > 1$, or $|r| > 1$ and $mn = 1$.

If $d = \text{g.c.d.}(m, n) > 1$, add the relator a^d, obtaining $C_d * C$. This free product maps onto $C_d \wr C$, the wreath product of a cyclic group of order d by an infinite cyclic group C. But $C_d \wr C$ is metabelian, and its multiplicator $H_2(C_d \wr C)$ is not finitely generated (see [6], or [7, p. 435]). By the above remark, $H_2(G/G'')$ is not finitely generated either.

If $|r| > 1$ and $mn = 1$, add the relator t^r and argue correspondingly. (One also can reduce this subcase to the previous one by exchanging the roles of a and t.)

3. The Structure of G/G'' in Case (ii) or (iii)

3.1. In this section, we shall give a detailed investigation of the structure of G/G'' in Case (ii) or (iii). This will allow us later on to replace G/G'' by a quotient that is easier to deal with. As a by-product, it will turn out that, in Case (ii) or (iii), G/G'' is torsion-free of finite torsion-free rank.

3.2. Lemma A. *Let*

$$G = \langle a, t : (a^m)^{t^r} = a^n \rangle, \qquad mnr \neq 0.$$

Assume that m and n are relatively prime, and $mn \neq 1$. Let B denote the normal closure of a^{n-m} in G and let N denote the normal closure of a in G. Then

$$B \trianglelefteq G' \trianglelefteq N,$$

and the following statements are true:

(a) $N' \cap G'' \cdot B = G''$,

(b) $G'' \cdot B/G'' \cong N' \cdot B/N' \hookrightarrow N_{\mathrm{ab}}$, *and all three groups are isomorphic with*

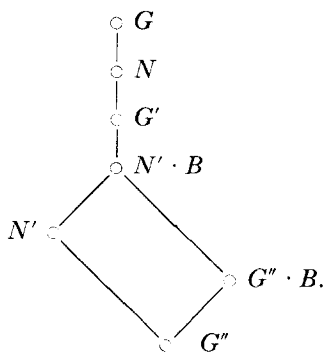
$$Q_{(m \cdot n)} \oplus \cdots \oplus Q_{(m \cdot n)}, \qquad (|r| \text{ summands}),$$

where $Q_{(m \cdot n)}$ denotes the additive subgroup of those rational numbers whose denominators can be written as powers of $m \cdot n$.

(c) $N'/G'' \cong N' \cdot B/G'' \cdot B \hookrightarrow G'/G'' \cdot B \cong (G'/B)_{\mathrm{ab}}$, *and all four groups are free abelian of rank*

$$(|r| - 1) \cdot (|m - n| - 1).$$

3.3. *Remarks.* The normal subgroups of G that are mentioned in the conclusion of Lemma A may be visualized by the following graph:



Statements (b) and (c) determine the structure of G'_{ab} qua abelian group. Indeed, G'_{ab} is an extension of $G'' \cdot B/G''$ by $G'/G'' \cdot B$ and $G'/G'' \cdot B$ is free abelian. Therefore,

$$G'_{\mathrm{ab}} \cong G'' \cdot B/G'' \oplus G'/G'' \cdot B \cong N_{\mathrm{ab}} \oplus N'/G''.$$

3.4. *Proof.* The quotient group G/N is infinite cyclic and generated by $t \cdot N$. Thus, by a standard result in the theory of one-relator groups (see, e.g., [10, Case 2 in the proof of Theorem 4.10, pp. 253–254]) the kernel N is generated by the elements

$$a_j = t^j a t^{-j}, \qquad (j \in \mathbb{Z}), \tag{3.1}$$

with defining relations

$$a_{j+r}^m = a_j^n, \qquad (j \in \mathbb{Z}). \tag{3.2}$$

From (3.1) and (3.2), a presentation for $N_{\mathrm{ab}} = N/N'$ is obtained by adding the commutativity relations

$$a_l \cdot a_k = a_k \cdot a_l, \qquad (l, k \in \mathbb{Z}). \tag{3.3}$$

This abelian group N_{ab} is clearly a direct sum of $|r|$ isomorphic copies A_i,

$$A_i = gp(a_{rj+i}, j \in \mathbb{Z}), \qquad (0 \leqslant i < |r|).$$

Each A_i is of torsion-free rank at most 1 and maps under the assignment

$$a_{rj+i} \mapsto (n/m)^{j \cdot \mathrm{sign}(r)},$$

onto $Q_{(m \cdot n)}$, the additive subgroup of those rationals whose denominators can be written as powers of $m \cdot n$. It follows that A_i is torsion-free. But a torsion-free abelian group of finite rank is hopfian, so that we end up with an isomorphism

$$N_{\mathrm{ab}} \simeq Q_{(m \cdot n)} \oplus \cdots \oplus Q_{(m \cdot n)}, \qquad (|r| \text{ summands}).$$

3.5. There remains to be seen what the kernel of the canonical projection

$$p : G/G'' \twoheadrightarrow G/N'$$

is like. Let B denote the normal closure of $b = a^{n-m}$ in G. Since $b = [t^r, a^m]$, the image $B \cdot G''/G''$ of B in G/G'' is abelian. The group $B \cdot G''/G''$ is, modulo G'', generated by the elements

$$b_j = t^j b t^{-j}, \qquad (j \in \mathbb{Z}),$$

and satisfies the relations

$$b_{j+r}^m \equiv b_j^n \quad \mathrm{mod}\ G'', \qquad (j \in \mathbb{Z}),$$

$$b_l \cdot b_k \equiv b_k \cdot b_l \quad \mathrm{mod}\ G'', \qquad (l, k \in \mathbb{Z}).$$

If we compare these generators and relations with (3.1), (3.2), and (3.3), we see that $B \cdot G''/G''$ is a quotient of N_{ab}. But the canonical image of $B \cdot G''/G''$ in N_{ab} is generated by the elements

$$t^j(a^{n-m})t^{-j} \cdot N' = (t^j a t^{-j})^{n-m} \cdot N', \qquad (j \in \mathbb{Z})$$

and thus, of rank $|r|$, since $m \neq n$. This implies that the map

$$B \cdot G''/G'' \to N/N'$$

is injective. Equivalently, $B \cdot G'' \cap N' = G''$. As a consequence, we obtain that

$$N'/G'' = N'/N' \cap B \cdot G'' \simeq N' \cdot BG''/B \cdot G'' = N' \cdot B/G'' \cdot B.$$

We next analyse N/B. A presentation is given by

$$\langle \{a_j\}_{j\in\mathbb{Z}} : a_{j+r}^m = a_j^{\ n},\ a_j^{n-m} (j \in \mathbb{Z})\rangle. \tag{3.4}$$

In (3.4) the generator a_j corresponds to the element $t^j a t^{-j} \cdot B$ of N/B (cf. the preceding subsection). Remember that m and n are relatively prime. The presentation (3.4), therefore, may be changed as indicated below:

$$\begin{aligned} N/B &= \langle \{a_j\}_{j\in\mathbb{Z}} : a_{j+r}^m = a_j^{\ n},\ a_j^{n-m}\rangle \\ &\cong \langle \{a_j\}_{j\in\mathbb{Z}} : a_{j+r} = a_j\ ,\ a_j^{n-m}\rangle \\ &\cong \langle \{a_j\}_{0\leqslant j<|r|} : a_j^{|n-m|}\rangle. \end{aligned} \tag{3.5}$$

Using (3.5), the structure of $N' \cdot B/G'' \cdot B$ can be determined. To that end, consider G'/B. Observe that $(G'/B)_{\text{ab}} = G'/G'' \cdot B$. Now, G'/B is normal in N/B. The quotient group $(N/B)/(G'/B)$ being cyclic of order $|n - m|$, it is not hard to present G'/B explicitly. Omitting the details, we only record that $(G'/B)_{\text{ab}}$ turns out to be free abelian of rank $(|r| - 1) \cdot (|n - m| - 1)$. Because $N/N' \cdot B = (N/B)_{\text{ab}}$ is a finite group (see (3.5)), in the chain of subgroups

$$N' \cdot B/G'' \cdot B \hookrightarrow G'/G'' \cdot B \hookrightarrow N/G'' \cdot B,$$

the first member has finite index in the third one, while the second has just been shown to be free abelian of rank $(|r| - 1) \cdot (|n - m| - 1)$. It follows that $N'/G'' \simeq N' \cdot B/G'' \cdot B$ is free abelian of the same rank.

4. The Proof of Lemma C

4.1. We begin with a reduction. The information given by Lemma A can be used to transform questions on G/G'' into questions on G/N', as is made precise in the next lemma.

Lemma B. *Let*

$$G = \langle a, t : (a^m)^{t^r} = a^n \rangle, \qquad mnr \neq 0.$$

Assume m and n are relatively prime and $mn \neq 1$. Let N denote the normal closure of a in G. Then, G/G'' is finitely related if an only if G/N' is finitely related. Similarly, $H_2(G/G'')$ is finitely generated if and only if $H_2(G/N')$ is finitely generated.

4.2. *Proof.* Every finitely generated metabelian group satisfies the maximal condition for normal subgroups (Hall [7, p. 430, Theorem 3]). Thus, N'/G'', the kernel of the canonical projection

$$p\colon G/G'' \twoheadrightarrow G/N', \tag{4.1}$$

is the normal closure of finitely many elements. It follows that G/N' is finitely related if G/G'' is so. For the converse, we use Lemma A. It says that N'/G'' is a free abelian group of finite rank, and hence, finitely related. Extensions of finitely related groups are finitely related (Hall [7, p. 426, Lemma 1]). Thus, G/G'' is finitely related if G/N' is so.

The second assertion is not much harder to prove. Assume first that $H_2(G/G'')$ is finitely generated. Since G/G'' is metabelian, it follows by the result of G. Baumslag quoted in Subsection 2.1 that the multiplicator of any image of G/G'' is finitely generated. Hence, in particular, $H_2(G/N')$ is finitely generated.

Conversely, assume that $H_2(G/N')$ is finitely generated. We shall invoke the spectral sequence theory of Lyndon and of Hochschild–Serre (see, e.g., [8, p. 303, Theorem 9.5]). This theory deals with an arbitrary extension

$$K \lhd Q \twoheadrightarrow Q/K$$

and among other things asserts that the multiplicator H_2Q of the middle group has a 4 term series whose three factors are isomorphic with suitable subquotients of the homology groups

$$H_0(Q/K, H_2K); \quad H_1(Q/K, H_1K); \quad H_2(Q/K), \tag{4.2}$$

respectively. In our case, K is the kernel of (4.1). By Lemma A, this kernel is a finitely generated abelian group. Thus, $H_2(N'/G'')$ is finitely generated, and so is its quotient group $H_0(G/N', H_2(N'/G''))$. As for the second term in (4.2) we remark that $Q/K = G/N'$ is generated by two elements. This implies that $H_1(G/N', H_1(N'/G''))$ is a subquotient of $H_1(N'/G'') \oplus H_1(N'/G'')$. The third term, namely, $H_2(G/N')$, is by hypothesis finitely generated. Thus, all three groups in (4.2) are finitely generated. It follows that $H_2(G/G'')$ is finitely generated.

4.3. We are now ready to deal with Case (ii). Assume $m = 1$, and $n \neq 1$ if $|r| > 1$. We have to find a finite presentation for G/G'', where G is the one-relator group

$$\langle a, t : a^{t^r} = a^n \rangle, \qquad nr \neq 0.$$

If $n = |r| = 1$, G is an abelian one-relator group. If $n \neq 1$, apply Lemma B. It states that G/G'' is finitely related if and only if G/N' is finitely related. Consider the group

$$G(0) = \langle a, t : a^{t^r} = a^n, [a, a^t], [a, a^{t^2}], \ldots, [a, a^{t^{|r|-1}}] \rangle. \tag{4.3}$$

It maps canonically onto G/N' and all the conjugates of a in $G(0)$ commute. Thus, G/N' coincides with $G(0)$, and (4.3) is a finite presentation of G/N'.

4.4. Our treatment of Case (iii) is based on Lemma C given below. This lemma also explains the role played by the group $G(0)$ in Case (ii) (see previous subsection).

LEMMA C. *Let*

$$G = \langle a, t : (a^m)^{t^r} = a^n \rangle, \qquad mnr \neq 0.$$

Assume m and n are relatively prime and $mn \neq 1$. Let N denote the normal closure of a in G. Then, the following statements are equivalent:

(a) $|m|$, *or* $|n| = 1$.

(b) $G(0) = \langle a, t: (a^m)^{t^r} = a^n, [a, a^t], \ldots, [a, a^{t^{|r|-1}}] \rangle$, *is metabelian.*

(c) G/N' *is finitely related.*

(d) G/G'' *is finitely related.*

4.5. *Proof.* Statements (c) and (d) are equivalent by Lemma B. The implications "(a) $\Rightarrow$ (b)" and "(b) $\Rightarrow$ (c)" have been established in Subsection 4.3. There remain the implications "not (a) $\Rightarrow$ not (b)" and "not (b) $\Rightarrow$ not (c)".

Suppose neither $|m|$ nor $|n|$ equals 1. Assume r is positive, and consider $N(0)$, the normal closure of a in $G(0)$. It may be presented as follows:

$$N(0) = \langle \{a_j\}_{j\in\mathbb{Z}} : a_{j+r}^m = a_j^n, [a_j, a_{j+k}]\ (0 < k < r) \rangle.$$

In analogy to the procedure used in dealing with one-relator groups, we introduce the basis group

$$B = \langle a_0, \ldots, a_r : a_r^m = a_0^n, [a_{j'}, a_{j''}]\ (0 \leqslant j' < j'' \leqslant r, j'' - j' < r)\rangle. \tag{4.4}$$

We assert that B is a subgroup of $N(0)$, and that B is not soluble. To see this, add the relator $[a_0, a_r]$ to the presentation (4.4), obtaining a free abelian group of rank r. It follows that the generators

$$a_0, \ldots, a_{r-1}$$

generate freely a free abelian subgroup of B, and that the same is true for the generators

$$a_1, \ldots, a_r.$$

By the general theory of free products with amalgamation, B can be written as

$$gp(a_0, \ldots, a_{r-1}) \underset{\psi}{*} gp(a_1, \ldots, a_r),$$

where the groups $gp(a_0, \ldots, a_{r-1})$ and $gp(a_1, \ldots, a_r)$ are free abelian on the displayed generators, and where ψ denotes the isomorphism given by

$$a_0^n \mapsto a_r^m,\ a_j \mapsto a_j, \qquad (0 < j < r).$$

By hypothesis, neither $|m|$ nor $|n|$ equals 1. It follows that B contains noncyclic free subgroups and so it is not soluble. As to the question why B is a subgroup of $N(0)$, we need merely observe that $N(0)$ is built up from copies of the basis group B in exactly the same fashion as if $G(0)$ were a one-relator group (cf. [10, p. 253–256, Case 2]). The only difference consists in that in the present case free abelian groups are amalgamated, whereas in the one-relator group case one amalgamates free groups.

4.6. We are left with an analysis of the implication "not (b) $\Rightarrow$ not (c)." Again, assume r is positive. Define for every natural number j a group $G(j)$ by setting

$$G(j) = \langle a, t : (a^m)^{t^r} = a^n, [a, a^{t^k}] \qquad (0 < k < (j+1)\cdot r)\rangle.$$

Note that the present $G(0)$ coincides with the group of the same name, introduced in Subsection 4.3. The canonical projections

$$p_j : G(j) \twoheadrightarrow G(j+1), \qquad (0 \leqslant j < \omega), \tag{4.5}$$

give rise to a diagram over the order type ω. Its colimit is precisely G/N'. By a result of Neumann [11, p. 124, Corollary (12)], we now face the following choice: Either the colimit G/N' coincides with $G(j^*)$ for all sufficiently large j^*, or G/N' is not finitely related. We shall see that all the $G(j)$'s are isomorphic. Thus, if $G(0)$ is not metabelian the first choice is ruled out, implying that G/N' is not finitely related.

It remains to see why all the $G(j)$'s are isomorphic. Let F be free on a and t, and let $R(j)$ denote the kernel of the presentation

$$F \twoheadrightarrow G(j).$$

Consider the power map

$$\sigma: F \to F;\ a \mapsto a^m,\ t \mapsto t.$$

We assert that σ maps $R(j+1)$ into $R(j)$ $(0 \leqslant j < \omega)$. The calculations are straightforward, and hence, omitted.

Remember that m and n are relatively prime. Thus, there exist integers M and N satisfying

$$1 = M \cdot m + N \cdot n.$$

Consider the element $(a^N)^{t^r} \cdot a^M$. Its mth power equals the element a modulo the normal closure of the two relators

$$(a^m)^{t^r} \cdot a^{-n} \qquad \text{and} \qquad [a, a^{t^r}].$$

We are thus lead to a root map

$$\rho: F \to F\colon a \mapsto (a^N)^{t^r} \cdot a^M, \qquad t \mapsto t.$$

We assert that ρ maps $R(j)$ into $R(j+1)$. The verification is again straightforward. Moreover, we have the congruences

$$\begin{aligned} \sigma \circ \rho(a) &\equiv a \qquad \text{modulo}((a^m)^{t^r} \cdot a^{-n}), \\ \sigma \circ \rho(t) &= t, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \rho \circ \sigma(a) &\equiv a \qquad \text{modulo}((a^m)^{t^r} \cdot a^{-n}), [a, a^{t^r}], \\ \rho \circ \sigma(t) &= t. \end{aligned} \tag{4.7}$$

Passing to the quotients $G(j) = F/R(j)$, the congruences (4.6) and (4.7) show that the root map ρ induces for every j an isomorphism

$$\rho(j) : G(j) \xrightarrow{\sim} G(j+1), \tag{4.8}$$

as desired.

4.7. *Remarks.* The exhibited isomorphisms may be combined with the projections p_j of (4.5) to the commutative diagram pictured below

$$\begin{tikzcd} G(0) \arrow[r, "\sim"] \arrow[d, two heads, "p_0"] & G(1) \arrow[r, "\sim"] \arrow[d, two heads, "p_1"] & G(2) \arrow[r, "\sim"] \arrow[d, two heads, "p_2"] & \cdots \arrow[r, "\sim"] & G(j) \arrow[r, "\sim"] \arrow[d, two heads, "p_j"] & \cdots \\ G(1) \arrow[r, "\sim"] & G(2) \arrow[r, "\sim"] & G(3) \arrow[r, "\sim"] & \cdots \arrow[r, "\sim"] & G(j+1) \arrow[r, "\sim"] & \cdots . \end{tikzcd} \tag{4.9}$$

Note that either all the surjections p_j are injective, or none of them is. Thus, if $G(0)$ is not metabelian, it is nonhopfian. An example of an ependomorphism that fails to be an automorphism is given by the composition ϑ: $G(0) \to G(0)$

$$\vartheta\colon G(0) \overset{p_0}{\twoheadrightarrow} G(1) \xleftarrow[\sim]{\rho(0)} G(0); \qquad a \mapsto a^m, \quad t \mapsto t.$$

For $r = 1$, the map ϑ was detected by Baumslag and Solitar [4].

We close with a word on the kernel of ϑ. As $\rho(0)$ is an isomorphism, $\ker \vartheta$ coincides with $\ker p_0$. Similarly, it follows from the diagram (4.9) that

$$\ker \underbrace{\vartheta \circ \vartheta \circ \cdots \circ \vartheta}_{j} = \ker(p_{j-1} \circ p_{j-2} \circ \cdots \circ p_1 \circ p_0). \tag{4.10}$$

Consequently, $\ker \vartheta^j$ is precisely the normal closure in $G(0)$ of the elements

$$[a, a^{t^k}](|r| \leqslant k < (j+1) \cdot |r|).$$

It is now clear that G/N' coincides with the limit group

$$G(0)\Big/\bigcup_{j<\omega} \ker \vartheta^j$$

(cf. Subsection 1.7).

5. The Multiplicator of G/G'' in Case (iii)

5.1. Our aim is to show that $H_2(G/G'')$ is finitely generated although G/G'' is not finitely related.

By Lemma B (Subsection 4.1) is sufficient (as well as necessary) to prove that $H_2(G/N')$ is finitely generated. The reason for passing from G/G'' to G/N' is the fact that G/N' is easier to deal with than G/G''.

5.2. We compute first $H_2(G(0))$. The multiplicator of a group Q, presented as F/S, is given by (5.1) (Schur–Hopf)

$$(S \cap F')/[S, F]. \tag{5.1}$$

The group $G(0)$ has the presentation

$$F/S = \langle a, t : (a^m)^{t^r} = a^n, [a, a^t], ..., [a, a^{t^{|r|-1}}] \rangle.$$

In Case (iii), m is different from n. Thus, no nontrivial power of the relator $(a^m)^{t^r} a^{-n}$ lies in the commutator subgroup F' of F. It follows that $S \cap F'/[S, F]$ is generated by the cosets

$$[a, a^t] \cdot [S, F], ..., [a, a^{t^{|r|-1}}] \cdot [S, F]. \tag{5.2}$$

(Note that the symbols "a" and "t" denote in (5.2) elements in the free group F, and not their corresponding images in $G(0)$.) For $0 < k < r$ (respectively, $0 > k > r$) the following congruences hold modulo $[S, F]$:

$$\begin{aligned}[a, a^{t^k}]^m &\equiv [a, (a^m)^{t^k}] \equiv [a, (a^n)^{t^{k-r}}] \equiv [a, (a^n)^{t^{k-r}}]^{t^{r-k}} \\ &\equiv [a^{t^{r-k}}, a^n] \equiv [a, a^{t^{r-k}}]^{-n}.\end{aligned} \tag{5.3}$$

Using the idea of (5.3) once more we obtain:

$$[a, a^{t^k}]^{m^2} \equiv [a, a^{t^{r-k}}]^{-nm} \equiv [a, a^{t^k}]^{n^2}. \tag{5.4}$$

It follows from (5.3) that $H_2(G(0))$ is a finite group of exponent $|m^2 - n^2|$. More precisely, $H_2(G(0))$ is a quotient of the group

$$(\mathbb{Z}_{|m^2-n^2|})^{(|r|-1)/2}, \qquad (r \text{ odd}),$$

respectively,

$$(\mathbb{Z}_{|m^2-n^2|})^{(|r|/2)-1} \oplus \mathbb{Z}_{|m-n|}, \qquad (r \text{ even}).$$

5.3. We study next the homomorphism

$$\vartheta: G(0) \to G(0); \ a \mapsto a^m, \ t \mapsto t$$

introduced in Subsection 4.7. The induced map

$$\vartheta_* : H_2(G(0)) \to H_2(G(0))$$

is multiplication by m^2. Since m and n are relatively prime, and since $H_2(G(0))$ is of exponent $|m^2 - n^2|$, ϑ_* is an isomorphism. In Subsection 4.7, the group G/N' has been shown to be the colimit of the diagram

$$G(0) \xrightarrow{\vartheta} G(0) \xrightarrow{\vartheta} G(0) \xrightarrow{\vartheta} \cdots$$

over the directed set ω. But the functor H_2 commutes with colimits over directed sets. Consequently,

$$H_2(G(0)) \simeq H_2(G/N').$$

We remark that the ideas underlying our computation of $H_2(G/N')$ have been exposed in Subsection 1.7.

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